

Coupling of Higher Spin Gauge Fields to a Scalar Field in AdS_{d+1} and their Holographic Images in the d -Dimensional Sigma Model

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ABSTRACT

The three-point functions of two scalar fields σ and the higher spin field $h^{(\ell)}$ of $HS(4)$ on the one side and of their proposed holographic images α and $\mathcal{J}^{(\ell)}$ of the minimal conformal $O(N)$ sigma model of dimension three on the other side are evaluated at leading perturbative order and compared in order to fix the coupling constant of $HS(4)$. This necessitates a careful analysis of the local current $\Psi^{(\ell)}$ to which $h^{(\ell)}$ couples in $HS(4)$ and which is bilinear in σ .

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1 Introduction

In any known model the AdS/CFT correspondence is an unproven hypothesis still. If such model is derived from string theory as the standard case of AdS_5 supergravity and $SYM_4(\mathcal{N}=4)$, supersymmetry permits geometric arguments based on representation theory that support AdS/CFT correspondence and these arguments look quite convincing indeed.

But in models of the type of higher spin gauge fields ($HS(d+1)$) there is no supersymmetry a priori and the correspondence can be proved only by dynamical calculations both in AdS_{d+1} and CFT_d cases. Since in these models perturbative expansions with small coupling constants are mapped on each other, such calculations are technically feasible and the holographic mapping is order by order. We shall start such calculation for $HS(4)$ and the 3-dimensional conformal $O(N)$ sigma model now.

We concentrate on three-point function of two scalar and one higher spin field

	AdS_4	CFT_3
Scalar	$\sigma(z)$	$\alpha(x)$
HSF	$h^{(\ell)}(z)$	$\mathcal{J}^{(\ell)}(x)$

where $\alpha(x)$ is the “auxiliary” or “Lagrange multiplier” field and $\mathcal{J}^{(\ell)}(x)$ an almost conserved current, which is a traceless symmetric tensor. In the sigma model case the coupling constant is $O(\frac{1}{\sqrt{N}})$. In the higher spin field theory the coupling constant for $\sigma\sigma h^{(\ell)}$ interaction is g_ℓ , so that we expect

$$g_\ell = C^{(\ell)} \frac{1}{\sqrt{N}} . \quad (1)$$

We determine $C^{(\ell)}$ first in an ad hoc wave function normalization such that

$$\langle \alpha(x) \alpha(0) \rangle_{CFT} = (x^2)^{-\beta} \quad (2)$$

$$\langle \sigma(z_1) \sigma(z_2) \rangle_{AdS} = (2\zeta)^{-\beta} F \left[\begin{matrix} \frac{1}{2}\beta, \frac{1}{2}(\beta+1) \\ \beta - \mu + 1 \end{matrix} ; \zeta^{-2} \right] \quad (3)$$

$$\zeta = \frac{(z_1^0)^2 + z_2^0)^2 + (\vec{z}_1 - \vec{z}_2)^2}{2z_1^0 z_2^0} , \quad \mu = \frac{1}{2}d , \quad (4)$$

so that (2) is obtained from (3) by a “simple” boundary limit

$$\lim_{z_1^0 \rightarrow 0} \lim_{z_2^0 \rightarrow 0} (z_1^0 z_2^0)^{-\beta} \langle \sigma(z_1) \sigma(z_2) \rangle_{AdS} = \langle \alpha(\vec{z}_1) \alpha(\vec{z}_2) \rangle_{CFT} . \quad (5)$$

The higher spin fields are assumed to be normalized in the same fashion. At the end we renormalize the higher spin field such that $C^{(\ell)}$ is replaced by one.

We shall treat two versions of the minimal $O(N)$ sigma model. In the “free” case we have as a scalar field

$$\alpha_f(x) = \frac{1}{\sqrt{2N}} \phi_i(x) \phi_i(x) , \quad (6)$$

where $\phi_i(x), i = 1, 2, \dots, N$ is the $O(N)$ vector and space-time scalar field normalized so that

$$\langle \phi_i(x) \phi_j(x) \rangle_{CFT} = (x^2)^{-\delta} \delta_{ij} \quad , \quad \delta = \mu - 1 \quad (7)$$

and (2) follows from (7) and (6) with

$$\beta_f = 2(\mu - 1) = d - 2 . \quad (8)$$

In the “interacting” sigma model we have an interaction

$$\mathbf{z}^{1/2} \int dx \phi_i(x) \phi_i(x) \alpha(x) \quad (9)$$

and the interaction constant \mathbf{z} is expanded

$$\mathbf{z} = \sum_{k=1}^{\infty} \frac{\mathbf{z}_k}{N^k} . \quad (10)$$

The “free” theory is unstable and by renormalization flow approaches the stable “interacting” theory. The conformal scalar field $\sigma(z)$ on AdS_{d+1} is massive (tachyonic due to conformal coupling with the AdS metric) and has two boundary values from the two roots of the dimension formula

$$\Delta = \mu \pm (\mu^2 + m^2)^{\frac{1}{2}} , \quad (11)$$

where for $d = 3$

$$m^2 = \begin{cases} -2 & \text{in the free case} \\ -2 + O(\frac{1}{N}) & \text{in the interacting case} \end{cases} \quad (12)$$

so that

$$\Delta(d=3) = \begin{cases} \beta_f = 1 & \text{from (8)} \\ \beta = 2 + O(\frac{1}{N}) & \text{from (9)} . \end{cases} \quad (13)$$

2 Currents coupled to (conformal) higher spin fields in AdS

We assume that the interaction of a spin ℓ gauge field $h^{(\ell)}$ and two scalar fields $\sigma(z)$ is local and mediated by a current $\Psi^{(\ell)}$

$$\int \frac{dz}{(z^0)^{d+1}} Tr \{ \Psi^{(\ell)}(z) h^{(\ell)}(z) \} . \quad (14)$$

$\Psi^{(\ell)}$ and $h^{(\ell)}$ are symmetric tensors of rank ℓ . If we postulate that the covariant divergence of $\Psi^{(\ell)}$ is a trace term, the interaction is gauge invariant. Namely a gauge transformation of $h^{(\ell)}$, being of the form (classical)

$$h^{(\ell)} \rightarrow h^{(\ell)} + \nabla \Lambda^{(\ell-1)} , \quad (15)$$

where $\Lambda^{(\ell-1)}$ is a symmetric traceless tensor and $\nabla \Lambda^{(\ell-1)}$ is symmetrized, leads to the zero gauge variation of (14)

$$Tr \{ \nabla \Psi^{(\ell)} \Lambda^{(\ell-1)} \} = 0 . \quad (16)$$

This consideration is in agreement with the so called "Fronsdal" theory [1] of higher spin with double-traceless gauge fields and currents. Truncation of this higher spin theory to the *conformal higher spin theory* can be observed if we consider the corresponding double-traceless current and gauge field as a sum of two traceless objects, namely

$$\Psi^{(\ell)} = J^{(\ell)} + g^{(2)} \psi^{(\ell-2)} , \quad (17)$$

where $J^{(\ell)}$ and $\psi^{(\ell-2)}$ are now the *traceless* tensors, $g^{(2)}$ is the $D = d + 1$ dimensional AdS metric and symmetrization is assumed. Following [2] we call $\psi^{(\ell-2)}$ the compensator field. It is easy to see that $\psi^{(\ell-2)}$ plays the role of the traceless trace of the double-traceless current $\Psi^{(\ell)}$ and has to decouple in the conformal limit of higher spin theory. In other words we will assume that at the d dimensional boundary $M_d = \partial AdS_D$, Ψ^ℓ behaves as a conformal tensor field. Now we will consider the general structure of conformal higher spin currents in the AdS_D space constructed from the conformally coupled scalar field $\sigma(z)$ with the corresponding on-shell condition

$$\square \sigma(z) = \nabla \cdot \nabla \sigma(z) = \frac{D(D-2)}{4L^2} \sigma(z) . \quad (18)$$

The tachyonic mass here (we use in this section the mainly minus signature of the AdS metric ¹) arises as a result of conformal coupling of the conformal

¹We will use AdS conformal flat metric, curvature and covariant derivatives comutation rules of the type

$$\begin{aligned} ds^2 &= g_{\mu\nu} dz^\mu dz^\nu = \frac{L^2}{(z^0)^2} \eta_{\mu\nu} dz^\mu dz^\nu , \quad \eta_{z^0 z^0} = -1, \sqrt{-g} = \frac{1}{(z^0)^{d+1}} , \\ [\nabla_\mu, \nabla_\nu] V_\lambda^\rho &= R_{\mu\nu\sigma}{}^\rho V_\lambda^\sigma - R_{\mu\nu\lambda}{}^\sigma V_\sigma^\rho , \\ R_{\mu\nu\lambda}{}^\rho &= -\frac{1}{(z^0)^2} (\eta_{\mu\lambda} \delta_\nu^\rho - \eta_{\nu\lambda} \delta_\mu^\rho) = -\frac{1}{L^2} (g_{\mu\lambda} \delta_\nu^\rho - g_{\nu\lambda} \delta_\mu^\rho) , \\ R_{\mu\nu} &= -\frac{D-1}{(z^0)^2} \eta_{\mu\nu} = -\frac{D-1}{L^2} g_{\mu\nu} , \quad R = -\frac{D(D-1)}{L^2} . \end{aligned} \quad (19)$$

scalar $\sigma(z)$ with the AdS curvature $S = \int d^D z \sqrt{-g} \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{D-2}{4(D-1)} R \sigma^2 \right)$. For the investigation of the conservation and tracelessness conditions for general spin ℓ symmetric conformal current $J_{\mu_1 \mu_2 \dots \mu_\ell}^{(\ell)}$ we contract it with the ℓ -fold tensor product of a vector a^μ and make the ansatz including a first curvature correction in contrast to the free flat case [3]

$$\begin{aligned} J^{(\ell)}(z; a) &= \frac{1}{2} \sum_{p=0}^{\ell} A_p (a \nabla)^{\ell-p} \sigma(z) (a \nabla)^p \sigma(z) \\ &+ \frac{a^2}{2} \sum_{p=1}^{\ell-1} B_p (a \nabla)^{\ell-p-1} \nabla_\mu \sigma(z) (a \nabla)^{p-1} \nabla^\mu \sigma(z) \\ &+ \frac{a^2}{2L^2} \sum_{p=1}^{\ell-1} C_p (a \nabla)^{\ell-p-1} \sigma(z) (a \nabla)^{p-1} \sigma(z) + O(a^4) + O\left(\frac{1}{L^4}\right), \end{aligned} \quad (20)$$

where $A_p = A_{\ell-p}$, $B_p = B_{\ell-p}$, $C_p = C_{\ell-p}$ and $A_0 = 1$. Now we try to define the set of unknown constants A_p , B_p and C_p using the current conservation condition

$$\nabla \cdot \partial_a J^{(\ell)}(z; a) = \nabla^\mu \frac{\partial}{\partial a^\mu} J^{(\ell)}(z; a) = 0 \quad (21)$$

and the tracelessness condition connected with the conformal nature of our scalar field $\sigma(z)$

$$\square_a J^{(\ell)}(z; a) = \frac{\partial^2}{\partial a_\mu \partial a^\mu} J^{(\ell)}(z; a) = 0. \quad (22)$$

Using the following basic relations

$$[\nabla_\mu, (a \nabla)^p] \sigma = \frac{p(p-1)}{2L^2} (a_\mu (a \nabla)^{p-1} \sigma - a^2 (a \nabla)^{p-2} \nabla_\mu \sigma), \quad (23)$$

$$\begin{aligned} [\nabla_\mu, (a \nabla)^p] \nabla_\nu \sigma &= \frac{p(p-1)}{2L^2} (a_\mu (a \nabla)^{p-1} \nabla_\nu \sigma - a^2 (a \nabla)^{p-2} \nabla_\mu \nabla_\nu \sigma) \\ &+ \frac{p}{L^2} (g_{\mu\nu} (a \nabla)^p \sigma - a_\nu (a \nabla)^{p-1} \nabla_\mu \sigma), \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial}{\partial a^\mu} (a \nabla)^p \sigma &= p (a \nabla)^{p-1} \nabla_\mu \sigma \\ &+ \frac{p(p-1)(p-2)}{6L^2} (a_\mu (a \nabla)^{p-2} \sigma - a^2 (a \nabla)^{p-3} \nabla_\mu \sigma), \end{aligned} \quad (25)$$

$$\begin{aligned} \nabla \cdot \frac{\partial}{\partial a} (a \nabla)^p \sigma &= \frac{1}{L^2} \left[\frac{1}{4} p D (D-2) \right. \\ &\left. + p(p-1) \left(D + \frac{2}{3} p - \frac{7}{3} \right) \right] (a \nabla)^{p-1} \sigma + O\left(\frac{1}{L^4}\right), \end{aligned} \quad (26)$$

$$\begin{aligned} \square_a (a \nabla)^p \sigma &= \frac{1}{L^2} \left[\frac{1}{4} p(p-1) D (D-2) \right. \\ &\left. + \frac{1}{3} p(p-1)(p-2)(p+2D-5) \right] (a \nabla)^{p-2} \sigma + O\left(\frac{1}{L^4}\right) \end{aligned} \quad (27)$$

we can derive recursion relations for A_p , B_p and C_p coming from conservation condition (21)

$$pA_p + (\ell - p + 1)A_{p-1} + 2B_p + 2B_{p-1} = 0, \quad (28)$$

$$s_3(p)A_{p+1} + s_2(p, \ell, D)A_p + s_2(\ell - p + 1, \ell, D)A_{p-1} + s_3(\ell - p + 1)A_{p-2} + 2C_p + 2C_{p-1} = 0, \quad (29)$$

$$s_2(p, \ell, D) = \frac{1}{4}pD(D-2) + p(p-1)(D + \frac{1}{2}\ell + \frac{1}{6}p - \frac{7}{3}), \quad (30)$$

$$s_3(p) = \frac{1}{6}(p+1)p(p-1). \quad (31)$$

The relation (28) relates A_p and B_p recursively as in the flat case [3]. The next relation (29) arises from the $\frac{1}{L^2}$ correction and relates recursively C_p and A_p coefficients from our ansatz (20). From the other side the tracelessness condition (22) gives us two further relations between these coefficients

$$B_p = -\frac{p(\ell - p)}{(D + 2\ell - 4)}A_p, \quad (32)$$

$$C_p = \frac{-1}{2(D + 2\ell - 4)}[s_t(p+1, \ell, D)A_{p+1} + s_t(\ell - p + 1, \ell, D)A_{p-1}], \quad (33)$$

$$s_t(p, \ell, D) = \frac{1}{4}p(p-1)D(D-2) + \frac{1}{3}p(p-1)(p-2)(\ell + 2D - 5). \quad (34)$$

Again the relation (32) is the same as in the flat case and leads the Eq. (28) to the recursion

$$A_p = -s_1(p, \ell, D)A_{p-1}, \quad (35)$$

$$s_1(p, \ell, D) = \frac{(\ell - p + 1)(2\ell - 2p + D - 2)}{p(D + 2p - 4)}. \quad (36)$$

From this we can obtain the same solution for the A_p coefficients [3] as in the flat case

$$A_p = (-1)^p \frac{\binom{\ell}{p} \binom{\ell+D-4}{p+\frac{D}{2}-2}}{\binom{\ell+D-4}{\frac{D}{2}-2}}. \quad (37)$$

For the important case $D = 4$ this formula simplifies to

$$A_p = (-1)^p \binom{\ell}{p}^2. \quad (38)$$

It means that if our ansatz (20) and our consideration for the $\frac{1}{L^2}$ correction are right, the recursion relation for the A_p coefficients obtained by substituting the C_p coefficients in (29) by those of the $\frac{1}{L^2}$ tracelessness condition (33) must be

consistent with (35). Indeed using (33) and (35) we can rewrite the relation (29) in the form

$$(29) = A_p s_f(p, \ell, D) + A_{p-1} s_f(\ell - p + 1, \ell, D) = 0 \quad (39)$$

$$\begin{aligned} s_f(p, \ell, D) &= \left[s_2(p, \ell, D) - \frac{s_t(p, \ell, D)}{D + 2\ell - 4} \right. \\ &\quad \left. - s_1(p + 1, \ell, D) \left(s_3(p) - \frac{s_t(p + 1, \ell, D)}{D + 2\ell - 4} \right) \right] \\ &= \frac{(\ell + D - 3)(2\ell + D - 2)p(D + 2p - 4)}{4(D + 2\ell - 4)}. \end{aligned} \quad (40)$$

It is easy to see that the relation (39) coincides with (35) because

$$\frac{s_f(p, \ell, D)}{s_f(\ell - p + 1, \ell, D)} = s_1(p, \ell, d). \quad (41)$$

So we obtain a result that the structure of the conformal higher spin currents constructed from the conformal coupled scalar field in the fixed AdS background remains the same as in the free flat space case. We prove that our ansatz with $\frac{1}{L^2}$ correction connected with the difference between the traces in flat and AdS case does not violate the conservation condition (recursion relation (35)) for the coefficients A_p if they obey the tracelessness condition (33) for the currents. It means that the traceless conserved higher spin current constructed from conformal scalar field in AdS can be obtained from the flat space expression replacing usual derivatives with covariant ones and adding corresponding curvature corrections to the expression for the traces. For completeness we present in the Appendix an explicit derivation of the conformal conserved current in the case $\ell = 4, D = 4$ in all orders of $\frac{1}{L^2}$.

This phenomenon we can explain now in the following way: The conformal group for D -dimensional flat (with $SO(D - 1, 1)$ isometry) and AdS space (with $SO(D - 1, 2)$ isometry) is the same $-SO(D, 2)$ ². So we can say that the conformal primaries or the traceless conserved currents are the same due to the $\frac{1}{L^{2k}}$ corrections. But these originate from the curvature corrections to the flat space equation of motion and noncommutativeness of the covariant derivatives. Then because all currents are *traceless* we get the cancellation of all $\frac{1}{L^{2k}}$ accompanying terms coming from these two sources of deformation of the flat case relations in the conservation condition (21).

Now we will fix the coefficients A_p from the *CFT* consideration. We assume that on the boundary ∂AdS_{d+1} , $\Psi^{(\ell)}$ behaves as a conformal tensor field (the trace is decoupled). Moreover this conformal tensor must be local bilinear in $\alpha(x)$ of rank ℓ and of dimension

$$2\beta + \ell + O\left(\frac{1}{N}\right). \quad (42)$$

²Note that this is about conformal group of AdS space-not boundary

For this purpose we evaluate the 3-point function

$$\langle \alpha(x_1) \alpha(x_2) \frac{1}{2} \sum_{p=0}^{\ell} A_p \langle a \cdot \partial \rangle^p \alpha(x_3) \langle a \cdot \partial \rangle^{\ell-p} \alpha(x_3) \rangle_{CFT_3}, \quad (43)$$

where $\langle a \cdot \partial \rangle = a^i \partial_i$, $i = 1, 2, 3$.

From the propagator (2) for $\alpha(x)$ we obtain for (43)

$$2^\ell \sum_{p=0}^{\ell} A_p (\beta)_p (\beta)_{\ell-p} \frac{(x_{13}^2 x_{23}^2)^{-\beta}}{x_{13}^{2p} x_{23}^{2(\ell-p)}} \langle a \cdot x_{13} \rangle^p \langle a \cdot x_{23} \rangle^{(\ell-p)} + \text{trace terms}, \quad (44)$$

where we define the Pochhammer symbols $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$.

As a 3-point function of a conformal tensor is unique up to normalization

$$\mathcal{C} (x_{13}^2 x_{23}^2)^{-\beta} \{ \langle a \cdot \xi \rangle^\ell + \text{trace terms} \}, \quad (45)$$

$$\xi^i = \frac{x_{13}^i}{x_{13}^2} - \frac{x_{23}^i}{x_{23}^2}, \quad (46)$$

it follows

$$A_p = \frac{\mathcal{C} (-1)^p \binom{\ell}{p}}{2^\ell (\beta)_p (\beta)_{\ell-p}}. \quad (47)$$

This expression, for $\beta = 1$, is in agreement with the previous one (38) obtained from AdS_4 consideration, if we will normalize in (45) $\mathcal{C} = 2^\ell \ell!$.

For $\beta = 2$ we have to change the constraints imposed in (20) to allow compatibility of (47) with (20). We propose to give up the condition of tracelessness so that the coefficients in (20) are determined from (28), (29) and (47). This implies a coupling of the first trace compensator field of $h^{(\ell)}$ to $\Psi^{(\ell)}$.

3 The 3-point function $\sigma \sigma h^{(\ell)}$ in AdS_{d+1} to first order

Based on the interaction (14) with the general form of ansatz (20) we calculate the AdS 3-point function

$$g_\ell^{-1} \langle \sigma(z_1) \sigma(z_2) h_{\mu_1 \dots \mu_\ell}^{(\ell)} \rangle (b_{\mu_i})_\otimes^\ell. \quad (48)$$

We will use from this section the notation of Euclidian AdS with the Euclidian scalar product $\langle \dots \rangle$ both in the boundary and in the bulk space.

The bulk-to boundary propagator of a scalar field of dimension β is

$$K_\beta(w, \vec{x})|_{\vec{x}=0} = \left(\frac{w_0}{w_0^2 + \vec{w}^2} \right)^\beta \quad (49)$$

and of a tensor field which is traceless symmetric of dimension λ

$$K_\lambda^{(\ell)}(w, \vec{x})|_{\vec{x}=0} = \frac{w_0^{\lambda-\ell}}{(w_0^2 + \vec{w}^2)^\lambda} \left[\langle a, R(w) \vec{b} \rangle^\ell - \text{traces} \right] , \quad (50)$$

where

$$R_{\mu\nu}(w) = \delta_{\mu\nu} - 2 \frac{w_\mu w_\nu}{w_0^2 + \vec{w}^2}, \quad (51)$$

$$\sum_{\nu=0}^d R_{\mu\nu}(w) R_{\nu\lambda}(w) = \delta_{\mu\lambda} . \quad (52)$$

This kernel $R_{\mu\nu}$ is connected with the Jacobian of the inversion

$$z_\mu \rightarrow z'_\mu = \frac{z_\mu}{z_0^2 + \vec{z}^2} . \quad (53)$$

So that

$$\frac{\partial w'_\mu}{\partial w_\nu} = \frac{1}{w_0^2 + \vec{w}^2} R_{\mu\nu}(w), \quad (54)$$

$$R_{\mu\nu}(w) = R_{\mu\nu}(w') . \quad (55)$$

To first order in the interaction we obtain (48) with all arguments $z_{1,2,3}$ taken to the boundary

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^{\ell} A_n \int \frac{dw}{w_0^{d+1}} (\nabla_{\mu_i})_{\otimes}^n K_\beta(w, \vec{x}_1) (\nabla_{\mu_j})_{\otimes}^{\ell-n} K_\beta(w, \vec{x}_2) \prod_{i=1}^{\ell} g^{\mu_i \nu_i}(w) \\ & \times K_\lambda^{(\ell)}(w, \vec{x}_3)_{\nu_1 \dots \nu_\ell, \sigma_1 \dots \sigma_\ell} \prod_{k=1}^{\ell} b_{\sigma_k} = \langle \alpha(\vec{x}_1) \alpha(\vec{x}_2) \mathcal{J}_{i_1 \dots i_\ell}^{(\ell)} \rangle \prod_{k=1}^{\ell} b_{i_k} , \end{aligned} \quad (56)$$

where the equality is based on AdS/CFT correspondence.

On this integral we apply a d -dimensional translation

$$\vec{w} \rightarrow \vec{w} + \vec{x} \quad (57)$$

and inversion (53) to

$$w \rightarrow w', \quad \vec{x}_i \rightarrow \vec{x}'_i, \quad i \in \{1, 2\} . \quad (58)$$

This gives by contracting the R 's and by (52)

$$(x_{13}^2 x_{23}^2)^{-\beta} \int \frac{dw'}{(w'_0)^{d+1}} (\nabla'_{\mu_i})_{\otimes}^n K_\beta(w' - \vec{x}'_{13}) (\nabla'_{\mu_i})_{\otimes}^{\ell-n} K_\beta(w' - \vec{x}'_{23}) (w'_0)^{\lambda+\ell} \prod_{i=1}^{\ell} b_{\mu_i} . \quad (59)$$

Since $b_0 = 0$ the ∇' can be applied to \vec{x}'_{13} , \vec{x}'_{23} respectively and it follows

$$(x_{13}^2 x_{23}^2)^{-\beta} (-1)^\ell \langle \vec{b} \cdot \vec{\partial}'_{13} \rangle^n \langle \vec{b} \cdot \vec{\partial}'_{23} \rangle^{\ell-n} I_{\beta, \beta, \lambda + \ell} \left(\vec{x}'_{13}, \vec{x}'_{23} \right), \quad (60)$$

where³

$$\begin{aligned} I_{\Delta_1, \Delta_2, \Delta_3}(\vec{x}_1, \vec{x}_2) &= \int \frac{dw}{w_0^{d+1}} K_{\Delta_1}(w - \vec{x}_1) K_{\Delta_2}(w - \vec{x}_2) w_0^{\Delta_3} \\ &= \frac{1}{2} \pi^\mu \Gamma(\Sigma - \mu) \left[\prod_{i=1}^3 \frac{\Gamma(\Sigma - \Delta_i)}{\Gamma(\Delta_i)} \right] |\vec{x}_1 - \vec{x}_2|^{-2(\Sigma - \Delta_3)} \end{aligned} \quad (61)$$

and

$$\mu = \frac{1}{2}d, \quad \Sigma = \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3). \quad (62)$$

In our case

$$\begin{aligned} \Delta_1 &= \Delta_2 = \beta, \quad \Delta_3 = \lambda + \ell, \\ \Sigma &= \beta + \delta + \ell, \quad \Sigma - \mu = \beta + \ell - 1, \quad (\delta \text{ as in (7)}) \\ \Sigma - \Delta_1 &= \Sigma - \Delta_2 = \delta + \ell, \quad \Sigma - \Delta_3 = \beta - \ell - \delta. \end{aligned} \quad (63)$$

As in (46) we introduce

$$\vec{\xi} = \vec{x}'_{13} - \vec{x}'_{23} = \frac{\vec{x}_{13}}{x_{13}^2} - \frac{\vec{x}_{23}^i}{x_{23}^2} \quad (64)$$

and use

$$\langle \vec{b} \cdot \vec{\partial}'_{13} \rangle^n \langle \vec{b} \cdot \vec{\partial}'_{23} \rangle^{\ell-n} (\xi^2)^{-\Delta} = (-1)^n 2^\ell (\Delta)_\ell (\xi^2)^{-\Delta-\ell} \left\{ \langle \vec{b} \cdot \vec{\xi} \rangle^\ell + \text{trace terms} \right\}. \quad (65)$$

This yields finally

$$\begin{aligned} g_\ell^{-1} \langle \alpha(\vec{x}_1) \alpha(\vec{x}_2) \mathcal{J}_{i_1 \dots i_\ell}^{(\ell)} \rangle_{AdS} \prod_{k=1}^\ell b_{i_k} \\ = \mathcal{N}_{\alpha\alpha\mathcal{J}}^{AdS} (x_{12}^2)^{\delta-\beta} (x_{13}^2 x_{23}^2)^{-\delta} \left\{ \langle \vec{b} \cdot \vec{\xi} \rangle^\ell + \text{trace terms} \right\}, \end{aligned} \quad (66)$$

$$\mathcal{N}_{\alpha\alpha\mathcal{J}}^{AdS} = 2^\ell \pi^\mu \left(\frac{\Gamma(\delta + \ell)}{\Gamma(\beta)} \right)^2 \frac{\Gamma(\beta + \ell - 1) \Gamma(\beta - \delta)}{\Gamma(2\delta + 2\ell)} \frac{1}{2} \sum_{n=0}^\ell (-1)^n A_n. \quad (67)$$

The last factor gives using (47) and omitting the normalization factor \mathcal{C}

$$\frac{1}{2} \sum_{n=0}^\ell (-1)^n A_n = \frac{1}{2^{\ell+1} (\beta)_\ell} {}_2F_1 \left[\begin{matrix} -\ell, -\beta - \ell + 1 \\ \beta \end{matrix}; 1 \right] = \frac{(2\beta + \ell - 1)_\ell}{2^{\ell+1} (\beta)_\ell (\beta)_\ell}. \quad (68)$$

³since ℓ is even $(-1)^\ell = +1$

4 The CFT current $\mathcal{J}^{(\ell)}$

The current $\mathcal{J}^{(\ell)}$ is traceless symmetric of rank ℓ and has conformal dimension

$$\lambda = 2\delta + \ell + O\left(\frac{1}{N}\right). \quad (69)$$

For $\ell = 2$ this is the energy-momentum tensor of the $O(N)$ conformal sigma model (up to normalization) with exact dimension $2\delta + \ell$. For $\ell > 2$ (even) the anomalous dimension in (69) has been calculated for the interacting theory in [4]. Neglecting the anomalous dimension, the currents are conserved. To this leading order the currents can be expressed bilinearly by the $O(N)$ vector field $\phi^i(x)$

$$\mathcal{J}^{(\ell)}(x; a) = \frac{1}{\sqrt{2N}} \sum_{r=0}^{[\ell/2]} \sum_{n=0}^{\ell-2r} \mathcal{A}_{rn}^{(\ell)} (a^2)^r \langle \vec{a} \cdot \vec{\partial} \rangle^n (\vec{\partial})_{\otimes}^r \phi^i(x) \langle \vec{a} \cdot \vec{\partial} \rangle^{\ell-n} (\vec{\partial})_{\otimes}^r \phi^i(x) \quad (70)$$

(so that $\mathcal{J}^{(0)}$ equals α_f in (6)). We normalize $\mathcal{A}_{rn}^{(\ell)}$ to

$$\mathcal{A}_{00}^{(\ell)} = 1 \quad (71)$$

and get

$$\mathcal{A}_{rn}^{(\ell)} = \frac{(-1)^n \ell!}{2^r r! n! (\ell - n - 2r)!} \frac{(\delta)_{\ell-r}}{(\delta)_{r+n} (\delta)_{\ell-n-r}}. \quad (72)$$

Its two point function is

$$\begin{aligned} \langle \mathcal{J}^{(\ell)}(x; a) \mathcal{J}^{(\ell)}(0; b) \rangle_{CFT} = \mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT} (x^2)^{-\lambda} \left\{ \left[2 \frac{\langle \vec{a} \cdot \vec{x} \rangle \langle \vec{b} \cdot \vec{x} \rangle}{x^2} - \langle \vec{a} \cdot \vec{b} \rangle \right]^\ell \right. \\ \left. + \text{trace terms} \right\}. \end{aligned} \quad (73)$$

By explicit evaluation of the l.h.s. we obtain

$$\mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT} = 2^\ell (\ell!)^2 \frac{(2\delta + \ell - 1)_\ell}{(\delta)_\ell}. \quad (74)$$

5 The CFT 3-point function of α , α and $\mathcal{J}^{(\ell)}$

In a free conformal $O(N)$ sigma model the 3-point function can be directly calculated by contraction of the free fields $\phi^i(x)$

$$\langle \alpha(x_1) \alpha(x_2) \mathcal{J}^{(\ell)}(x_3; a) \rangle_{CFT} = \frac{2^{\ell+2} (\delta)_\ell}{\sqrt{2N}} (x_{12}^2 x_{13}^2 x_{23}^2)^{-\delta} \left\{ \langle \vec{\xi} \cdot \vec{a} \rangle^\ell + \text{traces} \right\} \quad (75)$$

with $\vec{\xi}$ as in (46), (64). In the interacting case we also make the ansatz

$$\langle \alpha(x_1) \alpha(x_2) \mathcal{J}^{(\ell)}(x_3; a) \rangle_{CFT} = \frac{1}{\sqrt{2N}} \mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT} (x_{12}^2)^{\delta-\beta} (x_{13}^2 x_{23}^2)^{-\delta} \left\{ \langle \vec{\xi} \cdot \vec{a} \rangle^\ell + \text{traces} \right\}. \quad (76)$$

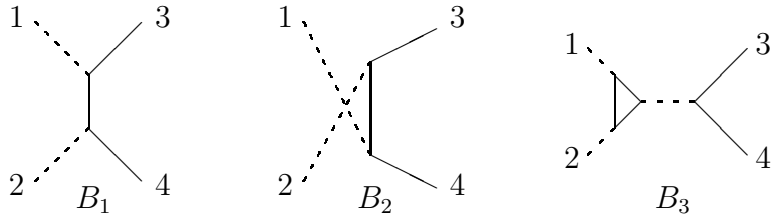
Contrary to AdS field theory this three-point function is obtained from a local interaction at second order.

In order to compute the proportionality factor $\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT}$ in (76) we start from the four-point function

$$\langle \alpha(x_1) \alpha(x_2) \phi_i(x_3) \phi_j(x_4) \rangle_{CFT}, \quad (77)$$

which we differentiate with respect to x_3, x_4 as prescribed by (70), then sum over $i = j$ and let finally $\vec{x}_3 - \vec{x}_4 \rightarrow 0$.

There are three graphical contributions



$$B_1, B_2, B_3 \quad (78)$$

at leading order $\frac{1}{N}$ to (77). Since the triangle subgraph of B_3 vanishes at $d = 3$ [6, 7] we neglect it here. B_2 is obtained by simple crossing of B_1 : $x_1 \leftrightarrow x_2$.

We define

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (79)$$

so that in the limit

$$\vec{x}_3 - \vec{x}_4 = \vec{x}_{34} \rightarrow 0 \quad (80)$$

$$u \rightarrow 0, \quad v \rightarrow 1. \quad (81)$$

A double power series expansion in u and $1 - v$ is correspondingly appropriate. The crossing $B_1 \rightarrow B_2$ implies

$$\begin{aligned} u &\rightarrow \frac{u}{v} \\ v &\rightarrow \frac{1}{v}. \end{aligned} \quad (82)$$

We remark also that all three graph have been calculated in another channel in [5], but we refrain from performing the nontrivial analytic continuation to our channel (defined by (81)).

The graph B_1 contributes a constant factor (for the notation see [7])

$$\mathbf{z}_1 v(\beta, \delta, \delta) v(1, \beta, 2\delta - \beta + 1)|_{\beta=2} = 2(2\delta - 1) \quad (83)$$

where $(\frac{u}{N})^{\frac{1}{2}}$ is the sigma model coupling constant. In addition B_1 gives a generalized hypergeometric function ($\beta = 2$)

$$(x_{13}^2 x_{24}^2)^{-\delta} (x_{12}^2)^{\delta-\beta} \sum_{n=0}^{\infty} \frac{u^n}{n!} \frac{[(\delta)_n]^2 n!}{(2\delta-1)_{2n} (\delta-1)_n} {}_2F_1 \left[\begin{matrix} \delta+n, \delta+n \\ 2\delta-1+2n \end{matrix}; 1-v \right] . \quad (84)$$

For the simple crossing (82) we can use the Euler identity ⁴

$$v^{-n} {}_2F_1 \left[\begin{matrix} \delta+n, \delta+n \\ 2\delta-1+2n \end{matrix}; 1-\frac{1}{v} \right] = v^{\delta} {}_2F_1 \left[\begin{matrix} \delta+n, \delta+n-1 \\ 2\delta-1+2n \end{matrix}; 1-v \right] . \quad (85)$$

For the covariant factor in (84) we obtain by crossing

$$(x_{23}^2 x_{14}^2)^{-\delta} v^{\delta} = (x_{13}^2 x_{24}^2)^{-\delta} . \quad (86)$$

Thus we have simply to add the two Gaussian functions (see [8] 9.137.17)

$$\begin{aligned} & {}_2F_1 \left[\begin{matrix} \delta+n, \delta+n \\ 2\delta-1+2n \end{matrix}; 1-v \right] + {}_2F_1 \left[\begin{matrix} \delta+n, \delta+n-1 \\ 2\delta-1+2n \end{matrix}; 1-v \right] \\ &= 2 {}_2F_1 \left[\begin{matrix} \delta+n, \delta+n-1 \\ 2\delta-2+2n \end{matrix}; 1-v \right] . \end{aligned} \quad (87)$$

Since the factor (83)

$$2(2\delta-1)|_{d=3} = 0 \quad (88)$$

vanishes, a non vanishing result necessitates a corresponding pole. Such pole is supplied by the coefficients of

$$\frac{u^n (1-v)^m}{n! m!} ,$$

whenever

$$2n + m \geq 2 . \quad (89)$$

Next we perform the differentiations

$$\sum_{\nu=0}^{\ell} \mathcal{A}_{0\nu}^{(\ell)} \langle \vec{a} \vec{\partial}_3 \rangle^{\nu} \langle \vec{a} \vec{\partial}_4 \rangle^{\ell-\nu} . \quad (90)$$

The powers u^n contain the factor $(x_{34}^2)^n$, which must be differentiated $2n$ times to be nonzero at $x_{34} = 0$. Doing this we obtain $(a^2)^n$ which contribute to the trace terms if $n > 0$. Thus we may assume $n = 0$ if we neglect the trace terms for the moment.

There remain terms proportional to

$$(x_{13}^2 x_{24}^2)^{-\delta} (1-v)^m = (x_{13}^2 x_{24}^2)^{-\delta-m} (x_{13}^2 x_{24}^2 - x_{14}^2 x_{23}^2)^m . \quad (91)$$

⁴2-term ${}_2F_1$ relation in [8] 9.131.1

From (76) we know that we must look for the factor

$$\langle \vec{a} \cdot \vec{\xi} \rangle^\ell = \left[\frac{\langle \vec{a} \cdot \vec{x}_{13} \rangle}{x_{13}^2} - \frac{\langle \vec{a} \cdot \vec{x}_{23} \rangle}{x_{23}^2} \right]^\ell . \quad (92)$$

For the identification of the normalization factor $\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT}$ it is sufficient to evaluate only

$$\left[\frac{\langle \vec{a} \cdot \vec{x}_{13} \rangle}{x_{13}^2} \right]^\ell . \quad (93)$$

To obtain it we may set

$$x_{23}^2 = x_{24}^2 \quad (94)$$

and differentiate only x_{13}, x_{14} with respect to x_3, x_4 before we set also in these terms $x_3 = x_4$. Thus we have to apply the differentiation (90) instead of (91) to

$$(x_{23}^2)^{-\delta} (x_{13}^2)^{-\delta-m} (x_{13}^2 - x_{14}^2)^m = (x_{23}^2)^{-\delta} \sum_{k=0}^m \binom{m}{k} (x_{13}^2)^{-\delta-k} (-x_{14}^2)^k . \quad (95)$$

In the limit $x_3 = x_4$ the result is

$$(x_{13}^2 x_{23}^2)^{-\delta} 2^\ell \left(\frac{\langle \vec{a} \cdot \vec{x}_{13} \rangle}{x_{13}^2} \right)^\ell \sum_{k=0}^{\ell-\nu} (-1)^{k-\ell+\nu} \binom{m}{k} \frac{k!}{(k-\ell+\nu)!} (\delta+k)_\nu . \quad (96)$$

Performing the summations (see the Appendix B) we get finally

$$\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT} = 4(2\delta-1) \frac{\ell!(\delta)_\ell}{(2\delta-1)_\ell} . \quad (97)$$

The loop of the ϕ^i -fields cancels the perturbative factor $\frac{1}{N}$.

6 Renormalization of the higher spin field

The normalization of the field $h^{(\ell)}$ has been fixed by the propagators (74), (50)

$$\lim_{w_0 \rightarrow 0} w_0^{-(\lambda-\ell)} \langle h^{(\ell)}(w) h^{(\ell)}(0) \rangle_{AdS} = (\mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT})^{-1} \langle \mathcal{J}^{(\ell)}(\vec{w}) \mathcal{J}^{(\ell)}(0) \rangle_{CFT} . \quad (98)$$

By AdS/CFT correspondence the CFT 3-point function (75), (76) where we insert the renormalized current $\mathcal{J}^{(\ell)}$

$$(\mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT})^{-\frac{1}{2}} \mathcal{J}^{(\ell)} \quad (99)$$

must equal the AdS 3-point function (66), (67)

$$\frac{1}{\sqrt{2N}} \frac{\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT}}{(\mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT})^{\frac{1}{2}}} = g_\ell \mathcal{N}_{\alpha\alpha\mathcal{J}}^{AdS} , \quad (100)$$

which implies

$$g_\ell = \frac{1}{\sqrt{N}} C^{(\ell)} , \quad (101)$$

$$C^{(\ell)} = \frac{1}{\sqrt{2}} \frac{\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT}}{(\mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT})^{\frac{1}{2}} \mathcal{N}_{\alpha\alpha\mathcal{J}}^{AdS}} . \quad (102)$$

Since we neglected a graph in the calculation of $\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT}$ which vanishes at $d = 3$, our result makes sense only at $d = 3$. We have

$$\mathcal{N}_{\alpha\alpha\mathcal{J}}^{CFT} = \begin{cases} 2^{\ell+2} (\frac{1}{2})_\ell & \text{(free)} \\ 4\ell (\frac{1}{2})_\ell & \text{(interacting)} \end{cases} , \quad (103)$$

$$\mathcal{N}_{\mathcal{J}\mathcal{J}}^{CFT} = 2^{3\ell-1} (\ell!)^2 , \quad (104)$$

$$\mathcal{N}_{\alpha\alpha\mathcal{J}}^{AdS} = \begin{cases} \frac{\pi^2}{2\ell} \left(\frac{\Gamma(\ell+\frac{1}{2})}{\ell!} \right)^2 & \text{(free)} \\ \frac{\pi^2}{2(\ell+2)} \left(\frac{\Gamma(\ell+\frac{1}{2})}{(\ell+1)!} \right)^2 & \text{(interacting)} \end{cases} , \quad (105)$$

implying

$$C^{(\ell)} = \begin{cases} \frac{2^{-\frac{1}{2}\ell+2}}{\pi^3} \ell \frac{\ell!}{(\frac{1}{2})_\ell} & \text{(free)} \\ \frac{4\sqrt{2}}{\pi^3} \ell(\ell+1)^2(\ell+2) \frac{\ell!}{(\frac{1}{2})_\ell} & \text{(interacting)} \end{cases} . \quad (106)$$

Finally we renormalize

$$H^{(\ell)}(z) = C^{(\ell)} h^{(\ell)}(z) . \quad (107)$$

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Appendix A

Spin 4 current in AdS_4 in details

We will define the traceless fourth rank tensor constructed from four dimensional on-shell scalar field $\sigma(z^\mu)$ in the following way

$$\begin{aligned} T_{\mu\nu\lambda\rho}^{\text{traceless}} &= T_{\mu\nu\lambda\rho} - \frac{3}{8} (g_{\mu(\nu} T_{\lambda\rho)} + T_{\mu(\nu} g_{\lambda\rho)}) + \frac{1}{16} g_{\mu(\nu} g_{\lambda\rho)} T , \\ T_{\mu\nu} &= T_{\alpha\mu\nu}^\alpha , \quad T = T_{\mu\nu}^{\mu\nu} . \end{aligned} \quad (A.1)$$

The conservation law which we will check below is

$$\nabla^\mu T_{\mu\nu\lambda\rho}^{\text{traceless}} = \nabla^\mu T_{\mu\nu\lambda\rho} - \frac{3}{8} (\nabla_{(\nu} T_{\lambda\rho)} + \nabla^\mu T_{\mu(\nu} g_{\lambda\rho)}) + \frac{1}{16} g_{\nu\lambda} \nabla_\rho T = 0 . \quad (A.2)$$

Finally we list here the most important on-shell relations (some of them are due to $\square\sigma(z) = \frac{2}{L^2}\sigma(z)$) we will use

$$[\square, \nabla_\mu] \sigma(z) = \frac{3}{L^2} \nabla_\mu \sigma(z) , \quad (\text{A.3})$$

$$[\nabla_\mu, \nabla_{(\nu\lambda\rho)}^3] \sigma(z) = \frac{3}{L^2} g_{\mu(\nu} \nabla_{\lambda\rho)}^2 \sigma(z) - \frac{3}{L^2} g_{\nu\lambda} \nabla_\rho \nabla_\mu \sigma(z) , \quad (\text{A.4})$$

$$\nabla_{(\mu\lambda\rho)}^3 \sigma(z) = \nabla_{(\lambda\rho)}^2 \nabla_\mu \sigma + \frac{1}{3L^2} g_{\mu(\rho} \nabla_{\lambda)} \sigma(z) - \frac{1}{L^2} g_{\lambda\rho} \nabla_\mu \sigma(z) , \quad (\text{A.5})$$

$$\nabla^\mu \nabla_{(\mu\lambda\rho)}^2 \sigma(z) = \frac{28}{3L^2} \nabla_{(\lambda\rho)}^2 \sigma(z) - \frac{8}{3L^4} g_{\lambda\rho} \sigma(z) , \quad (\text{A.6})$$

$$[\nabla_\mu, \nabla_{(\nu\lambda\rho)}^3] \nabla^\mu \sigma(z) = \frac{12}{L^2} \nabla_{(\nu\lambda\rho)}^3 \sigma(z) - \frac{9}{L^4} g_{\nu\lambda} \nabla_\rho \sigma(z) , \quad (\text{A.7})$$

$$g^{\lambda\rho} \nabla_{(\mu\lambda\rho)}^3 \sigma(z) = \frac{4}{L^2} \nabla_\rho \sigma(z) , \quad (\text{A.8})$$

$$g^{\lambda\rho} \nabla_{(\mu\nu\lambda\rho)}^4 \sigma(z) = \frac{20}{3L^2} \nabla_{(\mu\nu)}^2 \sigma(z) - \frac{4}{3L^4} \sigma(z) . \quad (\text{A.9})$$

Now we can construct directly the conserved spin 4 traceless current. First of all we note that from four derivatives we can construct only three bilinear combinations

$$T_{\mu\nu\lambda\rho}^{0,4} = \sigma \nabla_{(\mu} \nabla_\nu \nabla_\lambda \nabla_{\rho)} \sigma , \quad (\text{A.10})$$

$$T_{\mu\nu\lambda\rho}^{1,3} = \nabla_{(\mu} \sigma \nabla_\nu \nabla_\lambda \nabla_{\rho)} \sigma , \quad (\text{A.11})$$

$$T_{\mu\nu\lambda\rho}^{2,2} = \nabla_{(\mu} \nabla_\nu \sigma \nabla_\lambda \nabla_{\rho)} \sigma . \quad (\text{A.12})$$

For constructing the conserved (on-shell) combination of the traceless parts of these tensors we need first of all the on-shell value of their first and second traces

$$T_{\lambda\rho}^{0,4} = \frac{20}{3L^2} \sigma(z) \nabla_{(\mu\nu)}^2 \sigma(z) - \frac{4}{3L^4} \sigma^2(z) , \quad T^{0,4} = \frac{8}{L^4} \sigma^2(z) , \quad (\text{A.13})$$

$$T_{\lambda\rho}^{1,3} = \frac{1}{2} \nabla^\mu \sigma \nabla_{(\lambda\rho)}^2 \nabla_\mu \sigma + \frac{13}{6L^2} \nabla_\lambda \sigma \nabla_\rho \sigma - \frac{1}{6L^2} g_{\lambda\rho} \sigma \nabla_\mu \sigma , \quad T^{1,3} = \frac{4}{L^2} \nabla^\mu \sigma \nabla_\mu \sigma , \quad (\text{A.14})$$

$$T_{\lambda\rho}^{2,2} = \frac{2}{3} \nabla_\lambda \nabla^\mu \sigma \nabla_\rho \nabla_\mu \sigma + \frac{2}{3L^2} \sigma \nabla_{(\lambda\rho)}^2 \sigma , \quad T^{2,2} = \frac{2}{3} \nabla^{2(\mu\nu)} \sigma \nabla_{(\mu\nu)}^2 \sigma + \frac{4}{3L^4} \sigma^2 . \quad (\text{A.15})$$

Then introducing the following third rank symmetric tensor bilinear terms

$$\mathbf{A} = \nabla_{(\nu} \nabla^\mu \sigma \nabla_{\lambda\rho)}^2 \nabla_\mu \sigma , \quad \mathbf{a} = \nabla_{(\nu} \sigma \nabla_{\lambda\rho)}^2 \sigma , \quad (\text{A.16})$$

$$\mathbf{B} = g_{(\nu\lambda} \nabla_{\rho)} (\nabla^{2(\mu\nu)} \sigma \nabla_{(\mu\nu)}^2 \sigma) , \quad \mathbf{b} = g_{(\nu\lambda} \nabla_{\rho)} (\nabla^\mu \sigma \nabla_\mu \sigma) , \quad (\text{A.17})$$

$$\mathbf{C} = \nabla^\mu \sigma \nabla_{(\nu\lambda\rho)}^3 \nabla_\mu \sigma , \quad \mathbf{c} = \sigma \nabla_{(\nu\lambda\rho)}^3 \sigma , \quad \mathbf{d} = g_{(\nu\lambda} \nabla_{\rho)} (\sigma^2) \quad (\text{A.18})$$

and using (A.1)-(A.9), we obtain the following on-shell relations

$$\nabla^\mu T_{\mu\nu\lambda\rho}^{2,2\text{traceless}} = \frac{1}{2}\mathbf{A} - \frac{1}{12}\mathbf{B} + \frac{23}{4L^2}\mathbf{a} - \frac{9}{8L^2}\mathbf{b} - \frac{1}{4L^2}\mathbf{c} - \frac{19}{24L^4}\mathbf{d}, \quad (\text{A.19})$$

$$\nabla^\mu T_{\mu\nu\lambda\rho}^{1,3\text{traceless}} = \frac{9}{16}\mathbf{A} - \frac{3}{32}\mathbf{B} + \frac{1}{16}\mathbf{C} + \frac{51}{8L^2}\mathbf{a} - \frac{11}{8L^2}\mathbf{b} + \frac{1}{2L^2}\mathbf{c} - \frac{13}{8L^4}\mathbf{d}, \quad (\text{A.20})$$

$$\nabla^\mu T_{\mu\nu\lambda\rho}^{0,4\text{traceless}} = \mathbf{C} - \frac{3}{2L^2}\mathbf{a} - \frac{7}{4L^2}\mathbf{b} + \frac{25}{2L^2}\mathbf{c} - \frac{47}{4L^4}\mathbf{d}. \quad (\text{A.21})$$

Now we can see that the following unique combination of (A.10)-(A.12) is conserved

$$T_{\mu\nu\lambda\rho}^{s=4,\text{traceless}} = T_{\mu\nu\lambda\rho}^{2,2\text{traceless}} - \frac{8}{9}T_{\mu\nu\lambda\rho}^{1,3\text{traceless}} + \frac{1}{18}T_{\mu\nu\lambda\rho}^{0,4\text{traceless}}, \quad (\text{A.22})$$

$$\nabla^\mu T_{\mu\nu\lambda\rho}^{s=4,\text{traceless}} = 0. \quad (\text{A.23})$$

The expression (A.22) for the current is again in agreement with the flat space case general formula after a replacement of ordinary derivatives by covariant ones (compare the coefficients in (A.22) with the solution (38) and overall factor $\frac{1}{36}$).

Appendix B

We have to perform the triple summation

$$2(2\delta-1) \sum_{\nu=0}^{\ell} \binom{\ell}{\nu} \frac{(\delta)_\ell}{(\delta)_\nu (\delta)_{\ell-\nu}} \sum_{m=0}^{\ell} \frac{(\delta)_m (\delta)_{m-1}}{(2\delta-1)_{m-1} m!} \sum_{k=\ell-\nu}^m (-1)^k \frac{k!}{(k-\ell+\nu)!} \binom{m}{k} (\delta+k)_\nu, \quad (\text{B.1})$$

where the first term arises from $\mathcal{A}_{0\nu}^{(\ell)}$ in (90), the second term from the Gaussian hypergeometric series (87) and the last one from the expansion (96).

The k -summation is with

$$k = \ell - \nu + r$$

$$\sum_{r=0}^{m+\nu-\ell} \frac{\Gamma(-m+\ell-\nu+r)\Gamma(\delta+\ell+r)}{\Gamma(-m)\Gamma(\delta+\ell-\nu+r)r!} = \nu! m! \frac{(\delta)_\ell}{(\delta)_m} \frac{(-1)^m}{(\ell-m)!(\nu+m-\ell)!}. \quad (\text{B.2})$$

The m -summation is

$$\nu! (\delta)_\ell \sum_{m=\ell-\nu}^{\ell} \frac{(-1)^m}{(\ell-m)!(\nu+m-\ell)!} \frac{(\delta)_{m-1}}{(2\delta-1)_{m-1}}. \quad (\text{B.3})$$

After substitution of

$$m = \ell - \nu + s, \quad (\text{B.4})$$

it follows

$$= 2(-1)^{\ell-\nu} \frac{(\delta)_\ell (\delta-1)_{\ell-\nu}}{(2\delta-2)_{\ell-\nu}} \sum_{s=0}^{\nu} \frac{(-\nu)_s (\delta-1+\ell-\nu)_s}{s! (2\delta-2+\ell-\nu)_s} \quad (\text{B.5})$$

$$= 2(-1)^{\ell-\nu} \frac{(\delta)_\ell (\delta-1)_{\ell-\nu} (\delta-1)_\nu}{(2\delta-2)_\ell} . \quad (\text{B.6})$$

The ν -summation is

$$2(\delta)_\ell \sum_{\nu=0}^{\ell} \binom{\ell}{\nu} \frac{(\delta)_\ell}{(\delta)_\nu (\delta)_{\ell-\nu}} (-1)^{\ell-\nu} \frac{(\delta-1)_{\ell-\nu} (\delta-1)_\nu}{(2\delta-2)_\ell} \quad (\text{B.7})$$

$$= 2 \frac{[(\delta)_\ell]^2}{(2\delta-2)_\ell} \sum_{\nu=0}^{\ell} \frac{(-\ell)_\nu}{\nu!} \frac{(\delta-1)_\nu}{(\delta)_\nu} \frac{(\delta-1)_{\ell-\nu}}{(\delta)_{\ell-\nu}} . \quad (\text{B.8})$$

This sum can be decomposed into two ${}_2F_1(1)$ sums by

$$\frac{(\delta-1)_\nu}{(\delta)_\nu} \frac{(\delta-1)_{\ell-\nu}}{(\delta)_{\ell-\nu}} = \frac{\delta-1}{2\delta+\ell-2} \left[\frac{(\delta-1)_\nu}{(\delta)_\nu} + \frac{(\delta-1)_{\ell-\nu}}{(\delta)_{\ell-\nu}} \right] \quad (\text{B.9})$$

and we use for even ℓ

$$\frac{(-\ell)_\nu}{\nu!} = \frac{(-\ell)_{\ell-\nu}}{(\ell-\nu)!} \quad (\text{B.10})$$

to obtain

$$2 \frac{\ell! (\delta)_\ell}{(2\delta-1)_\ell} , \quad (\text{B.11})$$

which leads to (97) immediately.

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